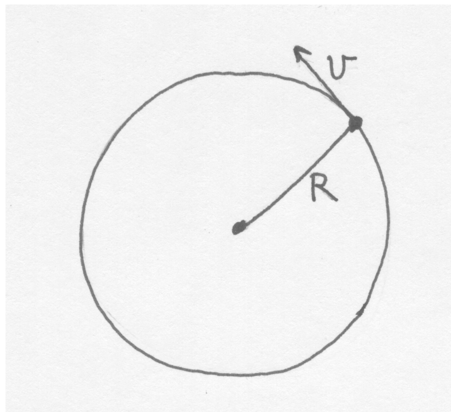
GFD I, 1/30/2012

Solutions to Problem Set #2

1) The physical setup looks like this:



The goal here is to show that the centripetal acceleration of the mass, which we assume to have magnitude v^2/R (in the fixed frame of reference), is consistent with the three terms on the right hand side of:

$$\frac{d\mathbf{u}_f}{dt} = \underbrace{\frac{d\mathbf{u}_r}{dt}}_{\text{(a)}} + \underbrace{2\mathbf{\Omega}\times\mathbf{u}_r}_{\text{(b)}} \underbrace{-\Omega^2\mathbf{R}}_{\text{(c)}}$$

In this problem, we have to consider the case when v is not equal to ΩR. So let's define the speed as consisting of two parts

$$v = V + v'$$

where we define the speed V as being $V = \Omega R$. We can also define the angular speed of the mass (in the fixed frame) as being

$$\omega = \Omega + \omega'$$

So now let's just compute the magnitude of the centripetal acceleration:

$$\frac{v^2}{R} = \frac{\left(V + v'\right)^2}{R} = \underbrace{\frac{V^2}{R}}_{(1)} + \underbrace{\frac{2Vv'}{R}}_{(2)} + \underbrace{\frac{v'^2}{R}}_{(3)}$$

Our task then is to prove that terms (a-c) are equivalent to terms (1-3) in the above equations. Now (for $v' > 0$) all the terms (a-c) are vectors pointing toward the center of the circle, so we need only consider their magnitude when doing the comparison. Using $V = \Omega R$ it is clear that (1) and (c) are equivalent in magnitude:

$$\left|-\Omega^2 \mathbf{R}\right| = \Omega^2 R = \frac{V^2}{R}$$

Also, it should be clear that the magnitude of $\mathbf{u}_r$ is equal to v', so we can easily show that terms (2) and (b) are equivalent:

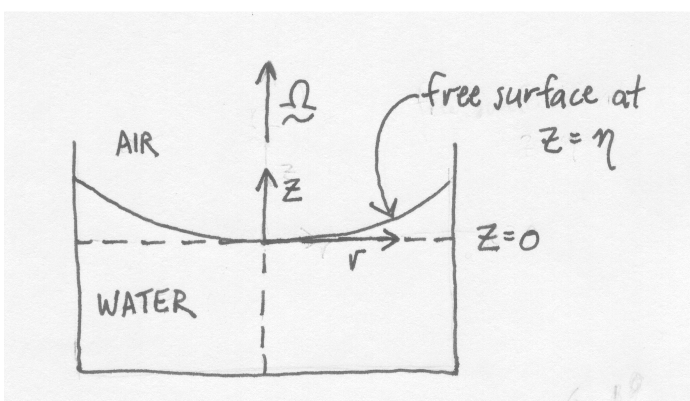
$$\left|2\mathbf{\Omega} \times \mathbf{u}_r\right| = 2\Omega v' = \frac{2Vv'}{R}$$

Finally, to complete the proof, we have to show that the magnitude of (a) is equal to (3). That this has to be true can be seen by realizing that $\mathbf{u}_r$ is a vector of magnitude v' which is turning in space with angular speed ω' (as observed from the rotating frame of reference) and so the magnitude of the time rate of change of this vector will be:

$$\left|\frac{d\mathbf{u}_r}{dt}\right| = v'\omega' = \frac{v'^2}{R}$$

This last bit of reasoning is just vector kinematics, not really physics, and it doesn't matter what frame of reference we do it in. Holton's textbook (1979) "An Introduction to Dynamic Meteorology," Section 1.5 discusses this, as do college physics textbooks. This final step completes the proof.

2) The physical situation for this problem looks like:



The term "solid body rotation" means that the fluid is not moving in a frame of reference rotating at the speed of the bowl, so the vertical force balance will be hydrostatic. Thus

$$\frac{\partial p}{\partial z} = -\rho g \qquad (+)$$

The radial force balance (here I am expressing it in the fixed frame of reference) will be between the centripetal acceleration (points toward the center) and the pressure gradient (also has to point toward the center, assuming we write the two on opposite sides of an equation). Mathematically this means:

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (++)$$

where v is the magnitude of the velocity in the "azimuthal" (around a circle) direction. Our job is to see if these two force balances may be used to predict the mathematical shape of the free surface. First it is helpful to realize that $v = \Omega r$, so the term on the left hand side of (++) can be written as $-v^2/r = -\Omega^2 r$ (note that the velocity increases with radius!). Then we need to figure out the radial pressure gradient. To do this, take a vertical integral of (+) from some arbitrary level z in the bowl, up to the free surface:

$$\int_z^\eta \frac{\partial p}{\partial z'} dz' = p_{atm} - p = -\int_z^\eta \rho g \, dz' = -\rho g(\eta - z)$$

$$\Rightarrow p = p_{atm} + \rho g(\eta - z)$$

where z' is a dummy variable of integration we use because z is one of the limits of the integral. Note that this expression for the pressure is valid at all depths. Then taking $\partial/\partial r$ of the equation for the pressure we find

$$\frac{\partial p}{\partial r} = \rho g \frac{\partial \eta}{\partial r} \qquad (+++)$$

because neither p_{atm} nor z are functions of r. Then using (+++) we may rewrite our radial force balance as

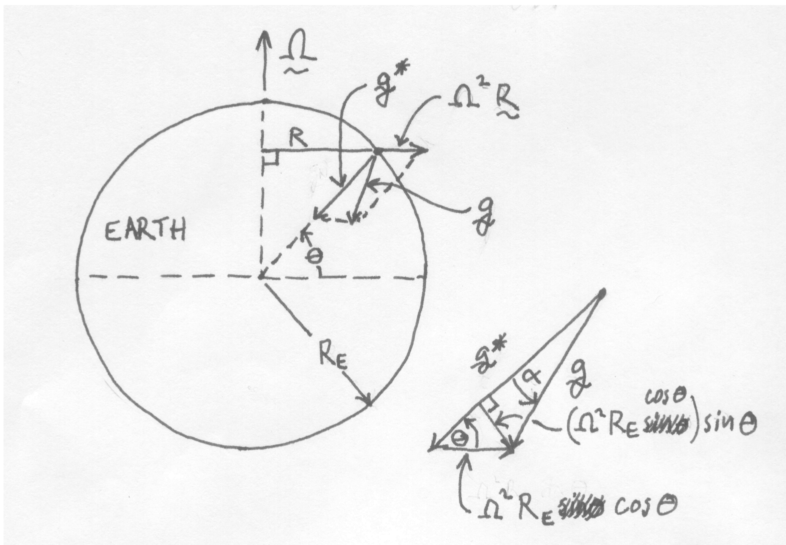
$$\frac{\partial \eta}{\partial r} = \frac{\Omega^2}{g} r$$

We may integrate this in r to find the expression for the free surface:

$$\eta = \frac{1}{2}\frac{\Omega^2}{g} r^2$$

(we have assumed that $\eta = 0$ at the center, but this is arbitrary, and does not change the nature of the solution). Since $\Omega = 2\pi/(2\pi \text{ s}) = 1 \text{ s}^{-1}$ the difference in the surface height from the edge to the center will be about 1/20th of a meter, or 5 cm.

3) For arbitrary latitude θ, the force balance between Newtonian gravity ($\mathbf{g}^*$ on the diagram below, and assumed in this problem to have a constant magnitude of 9.8 m s^{-2}) and the centrifugal acceleration on Earth looks like:



Note that we are in the rotating frame of reference here. On either pole the magnitude of the actual gravity vector $\mathbf{g}$ should just be the same as that of $\mathbf{g}^*$. On the equator it is reduced by an amount $-\Omega^2 R_E = -\left(7.3\times10^{-5} \text{ s}^{-1}\right)^2 (6371 \text{ km}) = 0.034 \text{ m s}^{-2}$, which is just 0.35% of gravity (or a fraction 0.0035 of gravity). Note that this argument neglects the change in magnitude of Newtonian gravity due to the ellipsoidal shape of the Earth. How big might this effect be compared to the centrifugal term calculated here?

The angular difference between the vectors $\mathbf{g}$ and $\mathbf{g}^*$ can be seen from the diagram to be given by:

$$\alpha = \frac{\Omega^2 R_E \cos(\theta)\sin(\theta)}{|\mathbf{g}|} \cong 0.1 \text{ degrees}$$

At $45°$ latitude.